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Existence Theorems for a Class of Nonlinear Operator Equations

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1. INTRODUCTION

Let X and Y be real Banach spaces, Ω an open bounded subset of X, and $\Gamma = (\{X_n\}, \{P_n\}, \{Y_n\}, \{Q_n\})$ an approximation scheme. Petryshyn [10–13] introduced the concept of A-proper mappings $T: \bar{\Omega} \to Y$ to construct solutions for the equation

$$Tx = f \tag{1}$$

as the strong limit of solutions of finite-dimensional equations approximating (1). Browder–Petryshyn [5, 6] introduced the concept of the degree for A-proper mappings $T: \bar{\Omega} \to Y$ which is a set-valued degree, having the basic properties of the Leray–Schauder degree. In [7] Fitzpatrick generalized the degree theory to a class of operators $T: \bar{\Omega} \to Y$ (Ω: bounded subset of X) which are the uniform limit of A-proper mappings. He showed that this degree also has properties analogous to those of the classical degree and indicated suitable applications of this degree in obtaining existence theorems to Eq. (1).

It is our object of the present paper to prove the existence of a solution to Eq. (1), where T is the uniform limit of A-proper mappings

$$T_t: \bar{\Omega}_D: = \bar{\Omega} \cap D \to Y.$$

Here D is an arbitrary and Ω an open bounded subset of X with closure $\bar{\Omega}$. In particular T must not be defined on the closure of some subset of X as in the cited papers. In Section 2 we prove two existence theorems to Eq. (1) using A-proper operators defined on $\bar{\Omega}_D$, homotopy arguments and degree theory. Some consequences of these theorems also are given. For mappings in Hilbert space we give in Section 3 sufficient conditions which ensure that the mapping is A proper.

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2. Existence Theorems to Eq. (1) for some Classes of Operators

Let X and Y be real Banach spaces. We shall use the symbols "$\rightarrow$" and "$\rightharpoonup$" to denote strong and weak convergence, respectively.

Let $D \subset X$ be arbitrary and $\Omega \subset X$ be open and bounded with closure $\bar{\Omega}$ and boundary $\partial\Omega$; then we define: $\Omega_D := \Omega \cap D$, $\bar{\Omega}_D := \bar{\Omega} \cap D$, and $\partial\Omega_D := \partial\Omega \cap D$. We remark that $\bar{\Omega}_D$ (respectively, $\partial\Omega_D$) is not the closure (respectively, boundary) of Ω_D.

Definition 1. A mapping $T: \bar{\Omega}_D \rightarrow Y$ is said to be quasi A proper if and only if for any sequence $\{x_n\} \subset \Omega_D$, such that $Tx_n \rightarrow f$ for some $f \in Y$, there exists an element $x \in \bar{\Omega}_D$ such that $Tx = f$.

Definition 2. By an (oriented) approximation scheme for mappings from X to Y we mean two sequences $\{X_n\}$ and $\{Y_n\}$ of oriented finite-dimensional subspace of X, respectively Y, such that $X_n \subset X_{n+1}$, respectively $Y_n \subset Y_{n+1}$, with $\dim X_n = \dim Y_n$ for all n and two sequences $\{P_n\}$ and $\{Q_n\}$ of bounded linear projections of X onto $\{X_n\}$, respectively Y, onto $\{Y_n\}$, such that $P_n x \rightarrow x$ for all $x \in X$ and $Q_n y \rightarrow y$ for all $y \in Y$ with $\| Q_n \| \leqslant K$ for all n with some constant $K > 0$.

Assumption 1. Let

$$\Omega_n := \Omega \cap X_n \subset \Omega_D, \qquad \bar{\Omega}_n := \bar{\Omega} \cap X_n \subset \bar{\Omega}_D$$

and

$$\partial\Omega_n := \partial\Omega \cap X_n \subset \partial\Omega_D \qquad \text{for all } n.$$

We note that Ω_n is an open bounded subset of X_n with closure $\bar{\Omega}_n$ and boundary $\partial\Omega_n$.

Definition 3. A mapping $T: \bar{\Omega}_D \rightarrow Y$ which satisfies Assumption 1 is said to be A proper with respect to the approximation scheme

$$\Gamma = (\{X_n\}, \{P_n\}, \{Y_n\}, \{Q_n\})$$

if and only if for any sequence $\{x_n\}$ such that $x_n \in \bar{\Omega}_n$ and $Q_n T x_n \rightarrow f$ for some $f \in Y$ there exists a subsequence $\{x_{n'}\}$ and an element $x \in \bar{\Omega}_D$ such that $x_{n'} \rightarrow x$ and $Tx = f$.

Lemma 1. *Let Assumption* 1 *be satisfied. Let* $0 \leqslant \epsilon_2 < \epsilon_1 < \infty$. *Suppose that, for all* $t \in [\epsilon_2, \epsilon_1]$, $T_t: \bar{\Omega}_D \rightarrow Y$ *is* A *proper with respect to the scheme* $(\{X_n\}, \{P_n\}, \{Y_n\}, \{Q_n\})$ *and*

$$T_t x \neq f$$

for all $x \in \partial\Omega_D$ *and all* $t \in [\epsilon_2, \epsilon_1]$. *Let* $T_t x$ *be continuous on* $[\epsilon_2, \epsilon_1]$ *uniformly for all* $x \in \bar{\Omega}_D$. *Then there exists an integer* $n_0 > 0$ *and a constant* $d > 0$ *such that*

$$\| Q_n T_t x_n - Q_n f \| \geqslant d$$

for all $x \in \partial\Omega_n$, *all* $t \in [\epsilon_2, \epsilon_1]$ *and each* $n \geqslant n_0$.

Proof. Suppose that the assertion of Lemma 1 is false, then there exists a sequence $\{n_j\}$ of positive integers with $n_j \to \infty$, a sequence $\{x_{n_j}: x_{n_j} \in \partial\Omega_{n_j}\}$, and a sequence $\{t_j\}$ with $t_j \in [\epsilon_2, \epsilon_1]$ such that

$$\| Q_{n_j} T_{t_j} x_{n_j} - Q_{n_j} f \| \to 0.$$

Because $t_j \in [\epsilon_2, \epsilon_1]$ there exists a subsequence (also denoted by t_j) and a number $t \in [\epsilon_2, \epsilon_1]$ such that $t_j \to t$. Thus we have by assumption of the lemma according to $x_{n_j} \in \partial\Omega_{n_j} \subset \partial\Omega_D \subset \bar{\Omega}_D$

$$\begin{aligned} \| Q_{n_j} T_t x_{n_j} - Q_{n_j} f \| &\leqslant \| Q_{n_j} T_t x_{n_j} - Q_{n_j} T_{t_j} x_{n_j} \| + \| Q_{n_j} T_{t_j} x_{n_j} - Q_{n_j} f \| \\ &\leqslant K \| T_t x_{n_j} - T_{t_j} x_{n_j} \| + \| Q_{n_j} T_{t_j} x_{n_j} - Q_{n_j} f \| \to 0 \end{aligned}$$

for $j \to \infty$, i.e., $Q_{n_j} T_t x_{n_j} \to f$. Because T_t is A proper with respect to the approximation scheme Γ there exists a subsequence $\{x_{n_j'}\}$ and an element $x \in \bar{\Omega}_D$ such that $x_{n_j'} \to x$ and $T_t x = f$. By $x_{n_j} \in \partial\Omega_{n_j} \subset \partial\Omega$ there follows $x \in \partial\Omega$, i.e., $x \in \partial\Omega \cap \bar{\Omega}_D = \partial\Omega_D$ contradicting the assumption of Lemma 1.

DEFINITION 4 (see [5, 6]). Let $T: \bar{\Omega}_D \to Y$ be an A proper mapping with respect to the scheme Γ and let $T_n := Q_n T |_{\bar{\Omega}_n}: \bar{\Omega}_n \to Q_n Y$ be continuous for each n. Let Z' be the set of all integers (positive, negative, and zero) together with $\{+\infty\}$ and $\{-\infty\}$. Furthermore let $f \notin T(\partial\Omega_D)$; then we define $\mathrm{Deg}_\Gamma(T, \Omega_D, f)$ to be the subset of Z' defined as follows:

(a) The integer m lies in $\mathrm{Deg}_\Gamma(T, \Omega_D, f)$ provided there exists a sequence $\{n_j\}$ such that $\deg(T_{n_j}, \Omega_{n_j}, Q_{n_j} f)$ is defined and equals m for all j;

(b) $+\infty$ $[-\infty] \in \mathrm{Deg}_\Gamma(T, \Omega_D, f)$ provided there exists a sequence $\{n_j\}$ such that $\deg(T_{n_j}, \Omega_{n_j}, Q_{n_j} f)$ is defined for all j and

$$\lim_j \deg(T_{n_j}, \Omega_{n_j}, Q_{n_j} f) = \infty[-\infty].$$

Remark. The degree $\deg(T_{n_j}, \Omega_{n_j}, Q_{n_j} f)$ used in Definition 4 is the classical Brouwer degree for mappings of oriented finite-dimensional Euclidean spaces of the same dimension.

We will now prove some existence theorems to Eq. (1) for mappings $T: \bar{\Omega}_D \to Y$.

THEOREM 1. (a) *Let the assumption of Lemma* 1 (*with* $\epsilon_2 = 0$, $\epsilon_1 = 1$) *be satisfied. Let* $Q_n T_t x$ *be a continuous mapping of* $[0, 1] \times \bar{\Omega}_n$ *into* $Q_n Y$ *for all* n.

(b) *Suppose that* $\mathrm{Dege}(T_1, \Omega_D, f) \neq \{0\}$. *Then there exists at least one* $x_0 \in \Omega_D$ *such that*

$$T_0 x_0 = f.$$

Proof. According to Lemma 1 there exists an integer $n_0 > 0$ and a constant $d > 0$ such that

$$\| Q_n T_t x - Q_n f \| \geqslant d$$

for all $x \in \partial \Omega_n$, all $t \in [0, 1]$, and each $n \geqslant n_0$. In virtue of

$$\mathrm{Deg}_\Gamma(T_1, \Omega_D, f) \neq \{0\}$$

there exists a sequence $\{n_k\}$ such that

$$\deg(Q_{n_k} T_1, \Omega_{n_k}, Q_{n_k} f) \neq 0.$$

Hence by the classical degree theory we have for $n_k \geqslant n_0$,

$$\deg(Q_{n_k} T_0, \Omega_{n_k}, Q_{n_k} f) = \deg(Q_{n_k} T_1, \Omega_{n_k}, Q_{n_k} f) \neq 0,$$

i.e., there exists $x_{n_k} \in \Omega_{n_k}$ such that $Q_{n_k} T_0 x_{n_k} = Q_{n_k} f$. For $k \to \infty$ we have $Q_{n_k} f \to f$. Therefore in virtue that $T_0: \bar{\Omega}_D \to Y$ is A proper, it follows the existence of a subsequence $\{x_{n_k'}\}$ and an element $x_0 \in \bar{\Omega}_D$ such that $x_{n_k'} \to x_0$ and $T_0 x_0 = f$. By assumption $x_0 \notin \partial \Omega_D$, i.e., $x_0 \in \Omega_D$, proving Theorem 1.

THEOREM 2. (a) *Let Assumption* 1 *be satisfied. Let* $T_t: \bar{\Omega}_D \to Y$ *be* A *proper with respect to the scheme* $(\{X_n\}, \{P_n\}, \{Y_n\}, \{Q_n\})$ *for all* $t \in (0, 1]$ *such that*

$$T_t x \neq f$$

for all $x \in \partial \Omega_D$. *Suppose that* $T_0: \bar{\Omega}_D \to Y$ *is quasi* A *proper. Let* $T_t x$ *be a continuous mapping of* $[0, 1]$ *into* Y *uniformly for all* $x \in \bar{\Omega}_D$, *and for all* n, *let* $Q_n T_t x$ *be a continuous mapping of* $[0, 1] \times \bar{\Omega}_n$ *into* $Q_n Y$.

(b) *Suppose that* $\mathrm{Deg}_\Gamma(T_1, \Omega_D, f) \neq \{0\}$. *Then there exists at least one element* $x_0 \in \bar{\Omega}_D$ *such that*

$$T_0 x_0 = f.$$

Proof. Let $\{t_k\} \subset [0, 1]$ be a monotonically decreasing sequence such that $t_k \to 0$. Then, for $t \in [t_k, 1]$ and $x \in \bar{\Omega}_D$, $T_t x$ satisfies the assumptions of Theorem 2. Therefore there exists for every k an element $x_k \in \Omega_D$ such that $T_{t_k} x_k = f$. Hence by

$$T_0 x_k = (T_0 x_k - T_{t_k} x_k) + f$$

and the uniform continuity of $T_t x$ with respect to $x \in \bar{\Omega}_D$ the right-hand side converges to f for $k \to \infty$. By our conditions on T_0 there exists an element $x_0 \in \bar{\Omega}_D$ such that $T_0 x_0 = f$, which proves Theorem 2.

PROPOSITION 1. *Let Assumption* 1 *be satisfied. Let* $T: \bar{\Omega}_D \to Y$ *be quasi A proper such that*

$$\| Tx - f \| \geqslant d$$

for all $x \in \partial\Omega_D$ *with some constant* $d > 0$. *Suppose that, for all* n, $Q_n T$ *is continuous from* $\bar{\Omega}_n$ *to* $Q_n Y$. *Let* $S: \bar{\Omega} \to Y$ *be continuous and bounded,*[1] *and for all* $t \in (0, 1]$, *let* $T_t := tS + T$ *be A proper with respect to scheme* Γ. *Suppose* $\mathrm{Deg}_\Gamma(T_1, \Omega_D, f) \neq \{0\}$. *Then there exists at least one element* $x_0 \in \bar{\Omega}_D$ *such that* $Tx_0 = f$.

Proof. We apply Theorem 2. It is sufficient to prove $T_t x \neq f$ for all $x \in \partial\Omega_D$ and all $t \in [0, 1]$. By the assumption of the Theorem it follows

$$\| T_t x - f \| \geqslant \| Tx - f \| - t \| Sx \| \geqslant d - \frac{d}{2} = \frac{d}{2}$$

for all $x \in \partial\Omega_D$ and all $t \in [0, 1]$.

PROPOSITION 2. *Let* $\bar{\Omega}_D$ *be symmetric such that* $0 \in \Omega_D$, *i.e., if* $x \in \bar{\Omega}_D$ *it follows that* $-x \in \bar{\Omega}_D$. *Suppose that Assumption* (a) *of Theorem* 1 *is satisfied with* $f = 0$. *Furthermore, let*

$$T_1(x) = -T_1(-x)$$

for all $x \in \partial\Omega_D$. *Then it exists an element* $x_0 \in \Omega_D$ *such that*

$$T_0 x_0 = 0.$$

Proof. We apply Theorem 1. It is sufficient to prove

$$\mathrm{Deg}_\Gamma(T_1, \Omega_D, 0) \neq \{0\}.$$

[1] We may assume without loss of generality that $\sup_{x \in \bar{\Omega}} \| Sx \| \leqslant d/2$.

By the conditions of the theorem it follows that $Q_n T_1(x) = - Q_n T_1(-x)$ for all $x \in \partial\Omega_n$ and all n. Furthermore, by Lemma 1 there exists an integer $n_0 > 0$ and a constant $d > 0$ such that

$$\| Q_n T_1(x) \| \geqslant d$$

for all $x \in \partial\Omega_n$ and all $n \geqslant n_0$. Hence by the classical Borsuk theory $\deg(Q_n T_1, \Omega_n, 0)$ is an odd integer; therefore $\mathrm{Deg}_\Gamma(T_1, \Omega_D, 0) \neq \{0\}$.

PROPOSITION 3. *Let $\bar{\Omega}_D$ be symmetric such that $0 \in \Omega_D$. Suppose that Assumption* (a) *of Theorem* 2 *is satisfied with $f = 0$. Furthermore, let*

$$T_1(x) = - T_1(-x)$$

for all $x \in \partial\Omega_D$. Then there exists an element $x_0 \in \bar{\Omega}_D$ such that

$$T_0 x_0 = 0.$$

Proof. We apply Theorem 2. It suffices to prove $\mathrm{Deg}_\Gamma(T_1, \Omega_D, 0) \neq \{0\}$ which follows as in the proof of Proposition 2.

Remark. The operators $T_t: \bar{\Omega}_D \to Y$ studied in this paper are defined on $\bar{\Omega}_D := \bar{\Omega} \cap D$, where Ω is a bounded open subset of X (with closure $\bar{\Omega}$) and D is an arbitrary subset of X (e.g., a dense linear subset of X), i.e. $\bar{\Omega}_D$ is in general not a closed subset of X. Theorems 1, and 2 and Propositions 1–3 therefore generalize in one sense results of Petryshyn [10–13], Browder–Petryshyn [5, 6] and Fitzpatrick [7]. Such theorems may be of great importance in studying differential equations.

3. A-PROPER AND QUASI-A-PROPER MAPPINGS

In this section we will give sufficient conditions on $T: \bar{\Omega}_D \to Y$ to be A proper, respectively quasi A proper, with respect to the approximation scheme Γ.

ASSUMPTION 2. Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and let $\{H_n\}$ be a sequence of closed linear finite-dimensional subspaces of H such that $H_n \subset H_{n+1}$ and $\bigcup_{n=1}^{\infty} H_n$ is dense in H. Let P_n be the projection of H onto H_n. Suppose that D is a linear subset of H such that $H_n \subset D$ for all n. Suppose that $\Omega := \{x \in H: \| x \| < r\}$ with some $r > 0$.

Remark. It follows by Assumption 2 that $(\{H_n\}, \{P_n\}, \{H_n\}, \{P_n\})$ is an approximation scheme for mappings from H to H. As in Section 2 we define Ω_D, $\bar{\Omega}_D$, and $\partial\Omega_D$.

DEFINITION 5. Let A be a mapping with domain $D(A) \subset H$ and range $R(A) \subset H$. Then A is said to be

(i) monotone, if for all $x, y \in D(A)$

$$(Ax - Ay, x - y) \geqslant 0;$$

(ii) strictly monotone, if there exists a constant $c > 0$ such that for all $x, y \in D(A)$

$$(Ax - Ay, x - y) \geqslant c \| x - y \|^2.$$

DEFINITION 6. An operator A with $D(A) \subset H$ and $R(A) \subset H$ is said to be m monotonic iff

(i) A is monotone;

(ii) $R(I + A) = H$.

Remark. By Definition 6(i) it follows that $(I + A)^{-1}$ exists on $R(I + A)$ and is Lipschitz continuous (see, e.g., Kato [9]).

ASSUMPTION 3 (A_1). Let $A_1: \bar{\Omega}_D \to H$ such that $A_1(x) = A_0(x, x)$ satisfying the following conditions: (i) for all $x \in D$, $A_0(\cdot, x): \bar{\Omega} \to H$ is continuous from the weak to the strong topology; (ii) for all $x \in \bar{\Omega}$, $A_0(x, \cdot): D \to H$ is m monotonic.

ASSUMPTION 4 (A_1). Let $A_1(x) = A_0(x, x)$ with $A_0: \bar{\Omega} \times D \to H$ such that $A_0(x_n, P_n v) \to A_0(x_0, v)$ for all $v \in D$ and all sequences $\{x_n\}$ satisfying $x_n \in \bar{\Omega}_n$ and $x_n \rightharpoonup x_0 \in \bar{\Omega}$.

THEOREM 3. *Let $T: = A_1 + A_2$ where Assumptions 2, 3(A_1), and 4(A_1) are satisfied. Suppose that, for all $x \in \bar{\Omega}_D$, $A_0(x, \cdot)$ is strictly monotone and $A_2: \bar{\Omega}_D \to H$ is compact and continuous. Then $T: \bar{\Omega}_D \to H$ is A proper with respect to $\Gamma = (\{H_n\}, \{P_n\}, \{H_n\}, \{P_n\})$.*

Proof. Assumption 1 is satisfied. Let $\{x_n\}$ be a sequence such that $x_n \in \Omega_n$ and $P_n T x_n \to f$ for some $f \in H$ then we obtain by the conditions on A_2 that $P_{n'} A_1 x_{n'} \to f_1$ for some subsequence n' (with suitable $f_1 \in H$). By the boundedness of $\{x_{n'}\}$ there exists a subsequence (also denoted by $\{x_{n'}\}$) and an element $x_0 \in \bar{\Omega}$ such that $x_{n'} \rightharpoonup x_0$. Let $v \in D$; then by the conditions on A_0,

$$\begin{aligned} 0 &\leqslant (A_0(x_{n'}, x_{n'}) - A_0(x_{n'}, P_{n'} v), x_{n'} - P_{n'} v) \\ &= (A_0(x_{n'}, x_{n'}) - A_0(x_{n'}, P_{n'} v), P_{n'}(x_{n'} - v)) \\ &= (P_{n'} A_1(x_{n'}) - P_{n'} A_0(x_{n'}, P_{n'} v), x_{n'} - v) \to (f_1 - A_0(x_0, v), x_0 - v) \end{aligned}$$

for $n' \to \infty$. Hence

$$(f_1 - A_0(x_0, v), x_0 - v) \geqslant 0$$

for all $v \in D$. Therefore

$$\begin{aligned} \| x_0 - v \|^2 &\leqslant (x_0 - v, x_0 - v) + (f_1 - A_0(x_0, v), x_0 - v) \\ &\leqslant (x_0 - v + f_1 - A_0(x_0, v), x_0 - v). \end{aligned}$$

By Assumption 3(ii) there exists a unique $v_0 \in D$ such that

$$v_0 + A_0(x_0, v_0) = x_0 + f_1 .$$

Hence $\| x_0 - v_0 \| = 0$, i.e., $x_0 \in \bar{\Omega} \cap D = \bar{\Omega}_D$ and

$$A_0(x_0, x_0) = f_1 .$$

By the conditions on A_0,

$$\begin{aligned} c \| x_{n'} - P_{n'}x_0 \|^2 &= (A_0(x_{n'}, x_{n'}) - A_0(x_{n'}, P_{n'}x_0), x_{n'} - P_{n'}x_0) \\ &\leqslant (P_{n'}A_0(x_{n'}, x_{n'}) - A_0(x_{n'}, P_{n'}x_0), x_{n'} - P_{n'}x_0) \to 0 \end{aligned}$$

for $n' \to \infty$, i.e., $x_{n'} \to x_0$. Hence we obtain by definition of T,

$$A_0(x_0, x_0) + A_2(x_0) = T(x_0) = f,$$

proving Theorem 3.

THEOREM 4. *Let $T: = A_1 + A_2$ such that the Assumptions 2 and 3(A_1) are satisfied and $A_2: \bar{\Omega}_D \to H$ is compact and weakly continuous. Then $T: \bar{\Omega}_D \to H$ is quasi A proper.*

Proof. The proof follows as in Theorem 3. Let $\{x_n\} \subset \bar{\Omega}_D$ such that $Tx_n \to f$ for some $f \in H$; then by the boundedness of $\{x_n\}$ and the condition on A_2 there exists a subsequence $\{x_{n'}\}$ and an element $x_0 \in \bar{\Omega}$ such that $x_{n'} \rightharpoonup x_0 \in \bar{\Omega}$ *and* $A_1x_{n'} \to f_1$ with suitable $f_1 \in H$. By the conditions on A_1 we have for all $v \in D$

$$0 \leqslant (A_0(x_{n'}, x_{n'}) - A_0(x_{n'}, v), x_{n'} - v) \to (f_1 - A_0(x_0, v), x_0 - v)$$

for $n' \to \infty$. Hence, as in the proof of Theorem 3, we obtain $x_0 \in \bar{\Omega}_D$ and $A_1(x_0) = f_1$. By the weak continuity of the mapping $A_2: \bar{\Omega}_D \to H$ the assertion of Theorem 4 follows.

PROPOSITION 4. *Let H be a real Hilbert space, which satisfies Assumption 2. For $t \in [0, 1]$, let $T_tx: = tx + B_1(x)$ such that B_1 satisfies Assumptions 3(B_1) and 4(B_1). Then, for $t \in (0, 1]$, T_t is A proper with respect to Γ and T_0 is weakly A proper.*

Proof. The second part of the proposition follows directly by Theorem 4. The first part follows by Theorem 3, by setting

$$A_0(x, y) := ty + B_0(x, y)$$

and the remark that a mapping $A: D \to H$ being m monotonic yields $tI + A$ (with $t > 0$) is m monotonic (see, e.g., Kato [9], Lemma 2.1) and strictly monotone.

DEFINITION 7. A mapping $A: \bar{\Omega} \to H$ is said to be of type (M) provided the following condition hold: if $x_n \in \bar{\Omega}$, $x_n \rightharpoonup x_0$, $Tx_n \rightharpoonup z$ and

$$\limsup_n (Tx_n, x_n) \leqslant (z, x_0),$$

then $Tx_0 = z$.

DEFINITION 8. A mapping $A: \bar{\Omega} \to H$ is said to satisfy condition $(S)^+$ provided that whenever $\{x_n\} \subset \bar{\Omega}$ is such that $x_n \rightharpoonup x_0$ and

$$\limsup_n (Tx_n, x_n - x_0) \leqslant 0,$$

then $x_n \to x_0$.

The difinition of a mapping of type (M) given above is slightly different then that given by Brezis [1]. Mappings with property $(S)^+$ have been introduced by Browder [3] in the study of nonlinear eigenvalue problems.

THEOREM 5. *Let $T: = A_1 + A_2$ such that assumptions 2, $3(A_1)$, and $4(A_1)$ are satisfied. Suppose that $A_2: \bar{\Omega} \to H$ is of type (M), bounded, and satisfies condition $(S)^+$. Then T is A proper with respect to $(\{H_n\}, \{P_n\}, \{H_n\}, \{P_n\})$.*

Proof. Let $\{x_n\}$ be a sequence such that $x_n \in \Omega_n$ and $P_n T x_n \to f$ for some $f \in H$; *then there exists a subsequence* $\{x_{n'}\}$ *and an element* $x_0 \in \bar{\Omega}$ *such that* $x_{n'} \rightharpoonup x_0$. Let $v \in D$; then by the conditions on A_1 we have

$$\begin{aligned} \mathscr{T}_{n'} := {} & (P_{n'} A_0(x_{n'}, x_{n'}) - A_0(x_{n'}, P_{n'} v) + P_{n'} A_2(x_{n'}), x_{n'} - v) \\ & \to (f_1 - A_0(x_0, v), x_0 - v). \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} \mathscr{T}_{n'} = {} & (A_0(x_{n'}, x_{n'}) - A_0(x_{n'}, P_{n'} v) + A_2(x_{n'}), x_{n'} - P_{n'} v) \\ & + (A_0(x_{n'}, P_{n'} v), v - P_{n'} v) \\ \geqslant {} & (A_2(x_{n'}), x_{n'} - v) + (A_2(x_{n'}), v - P_{n'} v) + (A_0(x_{n'}, P_{n'} v), v - P_{n'} v) \\ \geqslant {} & (A_2(x_{n'}), x_{n'} - v) - \| A_2(x_{n'}) \| \| v - P_{n'} v \| \\ & + (A_0(x_{n'}, P_{n'} v), v - P_{n'} v). \end{aligned}$$

Hence, by the boundedness of A_2, and the conditions on A_0,

$$\limsup_{n'} \mathcal{T}_{n'} \geqslant \limsup_{n'} (A_2(x_{n'}), x_{n'} - v).$$

Therefore we have for all $v \in D$,

$$\limsup_{n'} (A_2(x_{n'}), x_{n'} - v) \leqslant (f - A_0(x_0, v), x_0 - v).$$

There exists a subsequence (also denoted by n') such that $A_2(x_{n'}) \rightharpoonup z$. From the last inequality we obtain for all $v \in D$

$$\| x_0 - v \|^2 + \limsup_{n'} (A_2(x_{n'}), x_{n'} - v) - (z, x_0 - v)$$
$$\leqslant (f + x_0 - v - A_0(x_0, v) - z, x_0 - v).$$

By Assumption 3(ii) there exists a unique $v_0 \in D$ such that

$$v_0 + A_0(x_0, v_0) = f + x_0 - z.$$

Hence we have

$$\| x_0 - v_0 \|^2 + \limsup_{n'} (A_2(x_{n'}), x_{n'} - v_0) \leqslant (z, x_0 - v_0)$$

from which follows that

$$\limsup_{n'} (A_2(x_{n'}), x_{n'}) \leqslant (z, x_0).$$

Therefore, by the conditions on A_2, we have $A_2(x_0) = z$, i.e.,

$$\limsup_{n'} (A_2(x_{n'}), x_{n'}) \leqslant (A_2(x_0), x_0)$$

from which

$$\limsup_{n'} (A_2(x_{n'}), x_{n'} - x_0) = \limsup_{n'} (A_2(x_{n'}), x_{n'}) - (z, x_0)$$
$$\leqslant (A_2(x_0), x_0) - (A_2(x_0), x_0) = 0.$$

Hence, by the conditions on A_2, $x_{n'} \to x_0$. Therefore, by the above inequality, we obtain

$$\| x_0 - v_0 \|^2 + (z, x_0 - v_0) \leqslant (z, x_0 - v_0),$$

which implies $x_0 = v_0 \in \bar{\Omega}_D$. Furthermore, we obtain

$$T(x_0) = A_0(x_0, x_0) + A_2(x_0) = f,$$

proving Theorem 5.

Remark. The results of this section generalize in one sense results of Petryshyn on A proper mappings.

Combining the existence theorems of Section 1 in a suitable manner with the sufficient conditions on A proper mappings of Section 2, we obtain existence theorems to Eq. (1), improving in one sense results on the maximal monotone mappings studied in several papers (see, e.g., [2, 4, 14, 15]) and the results of Hess [8] on generalizations of maximal monotone mappings by using homotopy arguments. It should be remarked that the operators studied in this paper are single valued in contrast to the maximal monotone mappings which are multivalued.

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